Stochastic Catastrophe Theory Applications in the Social Sciences

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Abstract

Various phenomena in the social sciences show multistability and hysteresis, suggesting they can be formulated as stochastic dynamical systems. Catastrophe theory is a multifaceted and powerful mathematical tool to classify and study a certain class of smooth dynamical systems; however, its application to the social sciences has been controversial. This thesis introduces catastrophe theory at a conceptual level, discusses current limitations in social science applications, and examines the merit of currently published applications. We conclude that due to current methodological limitations and difficulty of model usage the application of stochastic catastrophe theory in the social sciences is a difficult, niche case.

Stochastic Catastrophe Theory Applications in the Social Sciences

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Introduction

Although linear models are simple to apply and grasp, they are incapable of modeling non-linear phenomena with multiple stable states and the transitions between them. Consider, for example, bipolar disorder. Bipolar disorder can be charaterized by the individual switching between behavioral hypomania (abnormally high energy and activity level) and depression. We can conceptualize the energy level as a continuous random variable where low energy (depression) and high energy (hypomania) are two stable states that are more distant from each other than in healthy individuals. At any given moment, the person in question can transition from hypomania to depression and vice versa, and then tend to stay at that activity / energy level for some time. As a result, a time-series of the energy level of a bipolar individual would result in a bimodal probability distribution. Multi-modality is an indication that the underlying phenomenon is driven by a non-linear stochastic process (Cobb, 1978), and it might be fruitful to model it as a dynamical system. A dynamical system is a system in which some function (generally a differential equation) describes the evolution and time-dependence of the some variables over time. Psychology has recently seen a movement towards conceptualizing discontinuous and multi-modal phenomena like depressive episodes as dynamical systems (Cramer et al., 2016; van de Leemput et al., 2014). Work has also been done to model and anticipate critical transitions using dynamical systems approaches, but challenges with definitions, data collection, and operationalization remain (Helmich et al., 2021). Therefore, a rigorous dynamical systems approach that encompasses such multi-modal densities and non-linear transitions could be a valuable addition to the psychologist's analytical toolkit.

Catastrophe Theory (CT) is a subfield of dynamical systems and a contender for such a tool. CT is a formal mathematical framework for dynamical systems that explains non-linear transitions, multimodal densities, skew, kurtosis and other features described below in this thesis. CT builds on a long line of thought originating from the work of Henri Poincaré, George Birkhoff and many others with the goal of studying the topology, stability, and bifurcations of critical points (equilibria). CT was formulated by (Thom, 1972, 1994) and popularized by Zeeman (1976) who introduced the idea that CT could be applied to the social, biological, and financial sciences. It was Zeeman who coined the term 'Catastrophe Theory' and illustrated captivating thought experiments, such as using CT to model transitions between war and peace, stock market bull runs and crashes, or aggression and fear behaviors in dogs. The intuitive simplicity and accessibility of Zeeman's expositions captured the eye of the public, so much so, that even Salvador Dali's final painting, 'The Swallow's Tail' is a direct recognition of the beauty of CT. The focus of this thesis is to investigate the track record of stochastic CT applications in real-world scenarios, with a particular focus on the cusp catastrophe. The cusp is interesting because it is the simplest of the catastrophes that includes both uni-stable and bi-stable behavior and therefore has the capacity to model sudden transitions.

However, Zeeman's examples were only conceptual, and academics were quick to begin ringing alarm bells - pointing out the unfulfilled heavy-handed promises, misuse of a deterministic theory in a stochastic context, the employment of poorly operationalized ad-hoc control variables, a methodology that lacks quantitative rigor, and excessive reliance on cloudy qualitative features for estimation (Rosser, 2007; Sussmann & Zahler, 1978). Since then, stochastic interpretations of CT have been formulated, but significant obstacles to quantitative model fitting and application in the behavioral sciences remain and are discussed below in this thesis.

Preliminaries on Dynamical Systems

Dynamical systems theory studies how the behavior of a real or artificial system evolves over time. Some examples of real-world dynamical systems are the swinging of a pendulum, population growth, the orbits of objects in the solar system, or a drop of ink dissolving into a glass of water. In order to study a dynamical system we need a way to model some quantity x of the system (for instance, the temperature of a cooling object) as a function of time t. One of the predominant ways to model dynamical systems uses *differential equations.*

$$\frac{dx}{dt} = f(x) \tag{1}$$

A differential equation allows us to relate the rate of change of a quantity with the quantity itself. These rates of change allow us to iteratively reconstruct the behavior of the system as a function of time $x(t)$ as shown in Fig 1. We say that a function $x = h(t)$ is a *solution* to the differential equation given that it satisfies the differential equation (Stewart, 2016). In some special cases it is possible to derive a closed-form expression for $x(t)$, but most often this is not the case. In fact, most differential equations cannot be solved *analytically*, but are often solved *numerically.*

Conceptually, dynamical systems can be broken down into two parts - the *phase space*, and the *dynamics*. The phase space is a collection of all states a system could possibly take on. Each point in the phase space represents a possible state. For instance, the phase space of a cooling object contains all the possible temperatures the object could possibly take on. The dynamics of a system are then the rules that take the current state of the system, and transform it into the state at the next unit in time. In the case of the cooling object, the dynamics take the current temperature, and output the temperature in the next unit of time. In other words, the dynamics are described by a function that maps points in the phase space to other points in the phase space.

Consider an example differential equation describing dynamics of a cooling (or heating) object:

$$\frac{dT}{dt} = -k(T - T_a) \tag{2}$$

where T is the temperature of the object, k is a constant determining the rate of heat transfer, and T_a is a control parameter describing the ambient temperature of the environment. This differential equation tells us that at each point in time the rate of temperature change for the object is proportional to the difference between the current object temperature, and the ambient temperature of the environment, scaled by a constant k.

Once an initial condition for the state is known or chosen, we can apply the dynamics to get the state in the next unit of time. We can repeat this process to get a *trajectory* - a path of system states given an initial condition. Systems which, given an initial condition, always produce the same trajectory are called *deterministic*. There is no randomness or variation in the process of mapping one state to the next, and so each simulation under the same initial condition will produce identical behavior each time. If we want to precisely know the system's state a thousand time units into the future, we simply apply the dynamics rule to the current state a thousand times. Figure 1 shows two example trajectories for an object cooling and heating.

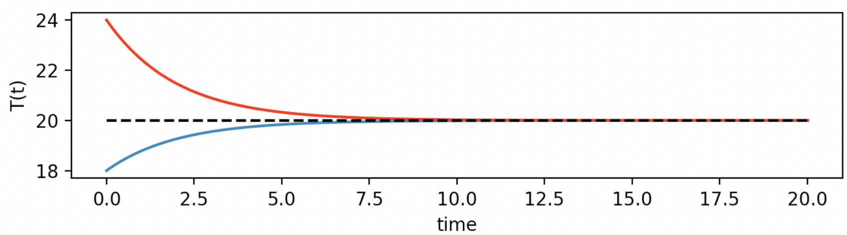

Figure 1

Time-series for the system $dT/dt = -k(T - T_a)$ describing an object cooling and heating. $k = 0.5$, $dt = 0.01$. The blue trajectory has initial condition $T = 18$, red trajectory has initial condition $T = 24$. The dotted line indicates the equilibrium state at $T = 20$.

Alternatively, systems that exhibit randomness in their dynamics will not produce identical trajectories for identical initial conditions and are said to be *stochastic*. Many phenomena observed in the real-world are stochastic processes. In this case, predicting how the system will behave in the future becomes much more difficult; we can only make probabilistic statements about future states at best. Figure 2 shows a single stochastic trajectory for an object warming to room temperature, and it's corresponding experimental *probability density function*. For simple systems like this, the probability density function, over long periods of time, is indicative of where the system state spends the most time, and therefore what states the system tends to evolve toward.

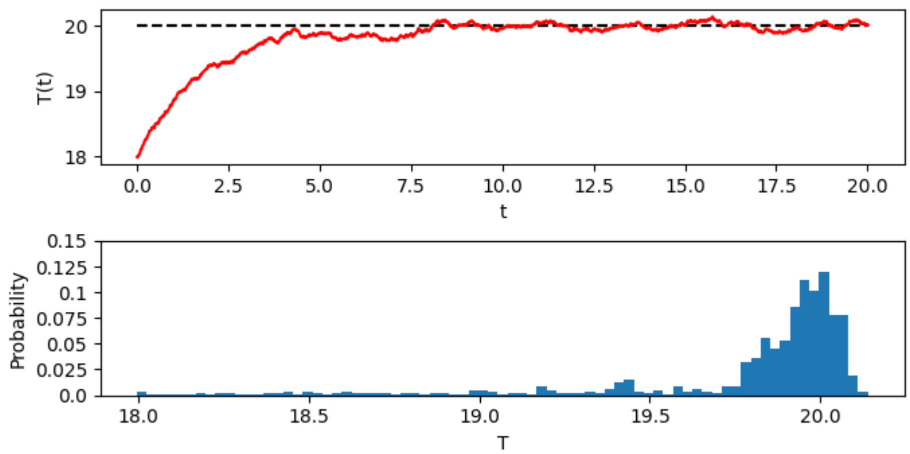

Figure 2

Top: Stochastic time-series for system $dT/dt = -k(T - T_a) + dW(t)$ *describing object cooling and heating. k = 0.5, dt = 0.01. Notice the dW(t) term which introduces randomness into the trajectory; W(t) is called a Wiener process and represents idealized one-dimensional Brownian motion. The red trajectory has initial condition T = 18. The dotted line indicates the equilibrium state at T = 20. Bottom: Histogram of T, scaled to approximate the probability density of the system.*

Modeling real-world systems in such an abstract way - as points being mapped to another in a space using rules, can lose a lot of the specialized contextual information of the real-world system. For instance, our object cooling or warming example system is blind to various details of the object and it's environment. This way we are discarding the non-essential specific details and real-world causes of the dynamics, but gain a generality that we could apply to other systems (e.g. other objects heating or cooling in other environments). This abstraction can be useful to study the core features and general behavior of the system and entire classes of systems. The *classification of dynamical systems* is important as it allows us to answer the question of whether two dynamical systems are equivalent. A bunch of billiard balls colliding on a pool table, a gas of particles in a small container, and a group of people moshing in a concert hall might seem like very different systems, governed by very different dynamics. Quantitatively they are different (in object size, shape, time scale, and so on), but qualitatively the dynamics of these systems are governed by the same rules for motion and collision.

An important feature that makes some dynamical systems qualitatively equivalent is their stability. Notice that in our example of a cooling object, the system tends towards 20 degrees - this is the *equilibrium* state. Equilibria are found where the derivative of the differential equation governing the dynamics equals zero.

$$\frac{dT}{dt} = -k(T - T_a) = 0 \tag{3}$$

Equilibria can be *stable* or *unstable*. The system state will evolve towards stable equilibria, and away from unstable equilibria. If an equilibrium point is reached, the system state will no longer change, unless it is externally perturbed away from the equilibrium. The form of the differential equation governing the dynamics of the system will therefore decide the type, location, and number of equilibrium states for that system. In our simple cooling object example, this is the state where the object's temperature equals the ambient temperature, and therefore the rate of temperature change over time is zero. If we vary the ambient temperature parameter T_a we can examine how the equilibrium of the system changes correspondingly. For such a simple linear system, the equilibrium will always simply be the ambient temperature of the surroundings as shown in Figure 1.

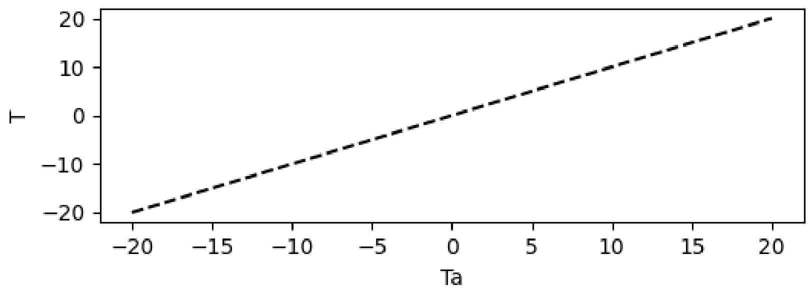

Figure 3

Equilibria for system $dT/dt = -k(T - T_a)$. $k = 0.5$. The dotted line indicates the equilibrium state as T_a is varied.

However, systems do not necessarily always have the same equilibrium structure at all times. In more complicated systems with non-linear dynamics, varying the parameters in the differential equation can cause not only *quantitative*, but also *qualitative* changes in stability. Equilibrium points can suddenly appear or disappear. A single equilibrium point can fork into many others. These qualitative changes to system stability and behavior are called *bifurcations*. Studying the equilibria and where they bifurcate in a dynamical system, can provide a global view of the system's behavior and be much more informative and generalizeable than just looking at individual trajectories. Moreover, bifurcations can reveal some surprising results and explain very complex behavior.

Consider, for example, a more complicated dynamical system with a single control parameter A.

$$\frac{dx}{dt} = -(x^3 - 0.5x - A) \tag{4}$$

Figure 4 depicts the equilibria of the system as a function of the parameter A. Gradual and continuous changes in the control variable A can result in the system state crossing a bifurcation point, and the behavior of the system changes as a consequence. If the system had begun at the lower line of equilibrium states and we gradually increase the parameter A, you'll notice that an additional two equilibria (one stable and one unstable) appear. If we continue increasing A, we reach yet another bifurcation point where the bottom stable and middle unstable equilibria disappear, and only the top stable equilibrium remains. Non-linear dynamical systems like this one exhibit *hysteresis*.

Hysteresis is the dependence of the system state on its history at previous time-points. Imagine again that the system initially begins on the lower stability sheet around $A = -0.25$; increasing A can lead to a slow increase in the value of the equilibrium. Even when a bifurcation occurs and a second stable equilibrium appears, the system will remain on the bottom equilibria line until it vanishes, and a catastrophic transition to the next available equilibrium follows. This is indicated by the blue arrow on the right. In order to return to the previous lower stability curve, the control variable A has to be reduced until the stable equilibria of the top sheet vanishes, and the system state plummets down to the nearest stable equilibrium at the bottom, as indicated by the red line. Therefore, a sudden transition can occur as a consequence of crossing a bifurcation point and stability being lost. A transition can also occur given a sufficient perturbation or noise that would put the state in sufficient proximity of another stable equilibrium.

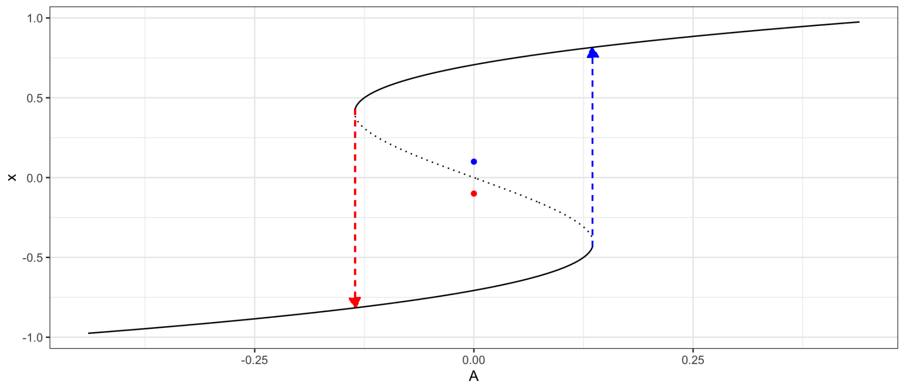

Figure 4

Equilibria for example system with dynamics $\frac{dx}{dt} = x^3 - 0.5x - A$*. Stable equilibria are lines in black; unstable equilibria are dotted gray. Hysteresis in this system can be demonstrated via the dashed red and blue paths - small and continuous changes in the control parameter (A) can lead to sudden and large transitions in state x. Where this transition occurs depends on previous system states. As we increase A, at around A = 0.13 we reach a bifurcation point; the local equilibrium curve no longer exists and a transition to the next stable equilibrium on the top fold follows. In order to reverse the transition and return to the bottom stability fold, notice that A would have to be reduced far more than for the transition upwards, to around A = -0.13.*

Systems like this can also exhibit very different behavior over time with just slight differences in initial conditions. Consider again the example dynamical system in Figure 4. The red and blue points in the middle of the graph represent example initial conditions, both just on either side of an unstable equilibrium. The values are very close together, however if we let these conditions evolve over time, the system's behavior will diverge to different values. This is something that becomes self-evident when studying a system's stability and bifurcations, but might not be clear from just looking at individual trajectories.

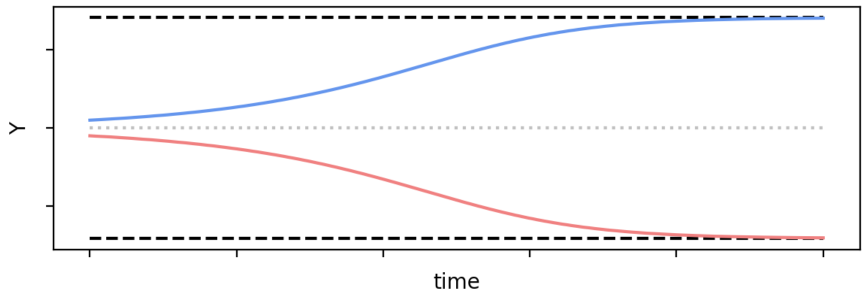

Figure 5

Deterministic time-evolution of dynamical system with identical control parameters and similar initial conditions shown in Figure 4. The system tends towards the nearest stable equilibrium points for that value of control parameter A. Note that the control parameter A remains static over time in this simulation.

Catastrophe Theory

Catastrophe theory (CT) deals with dynamical systems for which the equilibrium points can be defined as a minimum of a single smooth (differentiable everywhere), well-defined (unambiguous) potential function V. Such systems are called *gradient systems*. The potential function determines the direction and magnitude of the rate of change at each unit in time, and so decides the system dynamics entirely.

$$\frac{dx}{dt} = \frac{-dV(x;c)}{dx} \tag{5}$$

In the potential function x is a vector of system state (outcome) variables, and c is a vector of control variables (independent variables that determine system dynamics). The control variables are our inputs to the differential equation and determine the quantitative and qualitative changes in system stability. The form of this differential equation implies that the system state changes in the direction of decreasing potential at a rate proportional to the slope of the potential function. As before, we can find the equilibrium points where the potential function V is equal to zero.

As discussed above, classification of dynamical systems is important to study, understand, and generalize dynamical systems. Many dynamical systems have been be shown to be equivalent, even so however, there are far too many to classify them all. CT is remarkable in that it allows us to classify gradient systems universally; gradient systems that have two or fewer state variables, and five or fewer control variables can all be described by only seven *elementary catastrophes* or *unfoldings*. These elementary catastrophes are geometric structures that describe the equilibria and bifurcations of gradient systems. Figure 7 is an example of the cusp catastrophe - the equilibrium landscape for one state variable and two control variables. Each catastrophe comes with it's own associated potential function, which are described in Table 1 as presented in (Zeeman, 1976). Every smooth gradient system's bifurcation diagram with two control variables and one outcome variable can be found as a part or a section of the cusp catastrophe. CT therefore allows us to classify gradient systems with respect to the number and type of equilibrium points, and also study how the stability of the gradient system changes in response to control parameters being varied. These geometries of equilibria have been shown to be invariant to smooth (arbitrarily differentiable) and invertible one-to-one coordinate transformations (Berlinski, 1978). This means that two gradient systems are equivalent when one's potential function can be smoothly transformed into the other's, and their behavior will be captured by one of the seven catastrophes in the table below.

Table 1

The elementary catastrophes

Name	Potential	First Derivative	Control	Behavioral
Fold	$\frac{1}{3}x^3 - \alpha x$	$x^2 - \alpha$	1	1
Cusp	$\frac{1}{4}x^4 - \frac{1}{2}\beta x^2 - \alpha x$	$x^3 - \beta x - \alpha$	2	1
Swallowtail	$\frac{1}{5}x^5 - ax - \frac{1}{2}\beta x^2 - \frac{1}{3}cx^2$	$x^4 - a - bx - cx^2$	3	1
Butterfly	$\frac{1}{6}x^6 - ax - \frac{1}{2}bx^2 - \frac{1}{3}cx^3 - \frac{1}{4}dx^4$	$x^5 - a - bx - cx^2 - dx^3$	4	1
Hyperbolic-umbilic	$x^3 + y^3 + ax + by + cxy$	$3x^2 - y^2 + a + 2cx; -2xy + b + 2cy$	3	2
Elliptic-umbilic	$x^3 - xy^2 + ax + by + cx^2 + cy^2$	$3x^2 + a + cy; 3y^2 + b + cx$	3	2
Parabolic-umbilic	$x^2y + y^4 + ax + by + cx^2 + dy^2$	$2xy + a + 2cx; x^2 + 4y^3 + b + 2dy$	4	2

CT provides a mathematical basis for classifying gradient dynamical systems with respect to the number and type of equilibrium points, number of control variables, and number of behavioral variables.

The Fold Catastrophe

Consider the simplest of catastrophes; the fold has one control variable α and one behavioral outcome variable. It is an unfolding of the singularity x^3. The potential function for the fold catastrophe is:

$$V(x) = \frac{1}{3}x^3 - ax \tag{6}$$

and the derivate:

$$V'(x) = x^2 - a \tag{7}$$

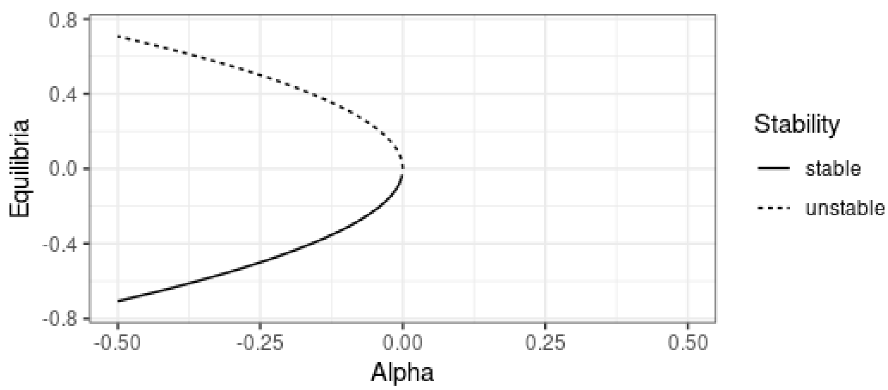

Figure 6

Fold equilibria plane

When $\alpha < 0$ the system has two stability points - one stable, and one unstable. At $\alpha = 0$ is where the stable and unstable equilibria meet and annihilate - the bifurcation point. At $\alpha > 0$ the system has no stable equilibrium. The fold catastrophe has been used to explain various physical phenomena, for instance, in analyzing tensile cracking and sliding rockburst instability (Wei et al., 2021).

The Cusp Catastrophe

The cusp catastrophe is an extension of the fold along a new control dimension, having one behavioral variable and two control variables - α and β. In the cusp catastrophe α is also often called the normal or asymmetry coordinate, as varying along it determines the direction and skew of the probability density function. Coordinate β is also often called the bifurcation or splitting coordinate as it determines the number of modes in the probability density function. The cusp catastrophe is shown in Figure 7. The canonical form of the potential function for the cusp is:

$$V(x; \alpha, \beta) = -(\alpha x + \frac{1}{2}\beta x^2 - \frac{1}{4}x^4) \tag{8}$$

and so the critical points can be found where

$$V'(x; \alpha, \beta) = -(\alpha + \beta x - x^3) = 0 \tag{9}$$

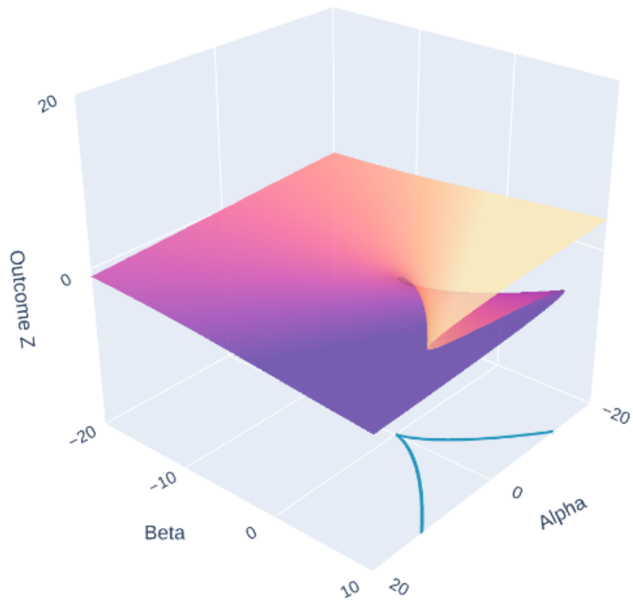

Figure 7

Cusp equilibria surface

As can be seen in Figure 7, the equilibrium surface of the cusp has one or three equilibria for all combinations of α and β. The plane at the bottom is called the control surface, and the blue lines indicate the boundary of the bifurcation region - where multiple equilibria meet and annihilate into one. This boundary is defined as *Cardan's Discriminant*:

$$\delta = 27\alpha^2 - 4\beta^3 \tag{10}$$

$\delta = 0$ marks the boundary of the bifurcation set. When $\delta > 0$ the potential function has three equilibria, and when $\delta < 0$ it has one.

All equilibria outside the bifurcation region are stable, whereas the maxima of the three equilibria within the bifurcation region are unstable - corresponding to the inner fold of the cusp. In a deterministic model any slight perturbation from this state would send the system towards one of the other equilibria. In a stochastic model (with noise) the unstable middle fold can be interpreted as the neighborhood the system state will spend the least time in.

The deterministic cusp time dynamics are described by the ordinary differential equation:

$$\frac{dx}{dt} = \alpha + \beta x - x^3 \tag{11}$$

Stochastic Cusp Catastrophe

Many real-world phenomena are not deterministic, but stochastic. In order to fit data with noise and random perturbations on to a cusp model an approach that accounts for this noise and perturbations is required.

There are two complementary approaches to fitting data to a catastrophe model. The first is an identification of qualitative flags proposed by Gilmore (1993). An example of such a flag is anomalous variance, where an increase in variance that occurs in the neighborhood of the bifurcation set, until a new plateau is reached. Finding such flags in the data can be an indication that the data may be described by a cusp (or a higher order catastrophe) model, but shouldn't be treated as conclusive evidence for it. The qualitative nature of these flags has rightfully been criticized as unrigorous; such flags should be at best used as heuristics to identify potentially suitable data. The second approach involves more rigorous quantitative fitting techniques - a brief overview of these are reviewed in this section.

Quantitative Fitting Techniques

Techniques to fit data to stochastic catastrophes are not trivial endeavors; known attempts are summarized below.

GEMCAT (Oliva et al., 1987) was introduced as a fitting technique that incorporated multivariate data and latent construct variables. GEMCAT, however, has a grievous issue - namely its inability to distinguish between stable and unstable equilibria (A. Hartelman, 1998).

Guastello's regression technique was an attempt to apply the cusp and butterfly catastrophes to drug addiction and work performance (Guastello, 1984) and academic performance (Guastello, 1987) data, among others. Alexander et al. (1992) have shown however, that Guastello's approach could not distinguish catastrophe data from random data. Guastello's technique also suffers from the same drawback of GEMCAT - it does not discriminate between the minima (stable equilibria) and maxima (unstable equilibria) of the potential function.

Therefore, both GEMCAT and Guastello's regression technique cannot be considered robust formulations of stochastic CT in their original form. The most robust and widely accepted stochastic formulation of CT without these issues was introduced by Cobb (1978, 1981, 2010) and (Cobb et al., 1983) across a series of papers.

Cobb's Method: To incorporate measurement error and noise in real-world data, Cobb used the results of Ito's stochastic calculus to include a white noise Wiener process dW with variance σ in the deterministic catastrophe. The system dynamics are then described by a stochastic differential equation:

$$dx = (-\frac{V'(x;c)}{dx})dt + \sigma dW, \tag{12}$$

where in the case of the cusp, V is defined by equation 8. The $-dV(x;c)/dx$ component is called the *drift* function, the $\sigma(x,t)$ is called the *diffusion* function, and W(t) is a Wiener process (idealized Brownian motion). The solution to equation 12 is a random variable, the probability density of which satisfies the Fokker-Planck equation (Pavliotis, 2014). The Fokker-Planck equation describes the evolution of the probability density function over time for a stochastic dynamical system. It can be shown then that with $-V'(x)$ and $\sigma(x,t)$ independent of x and t the associated theoretical stationary density is:

$$p(x) = \phi\frac{B}{\sigma^2}\exp(-\frac{2}{\sigma^2}V(x)), \tag{13}$$

where ϕ is a normalization constant. As can be seen in Figure 8, under the assumption of additive noise the modes of the stationary density function correspond to stable equilibria, whereas the antimodes correspond to unstable equilibria. Figure 9 shows a more global overview of how varying control parameters α β and diffusion σ influence the probability density.

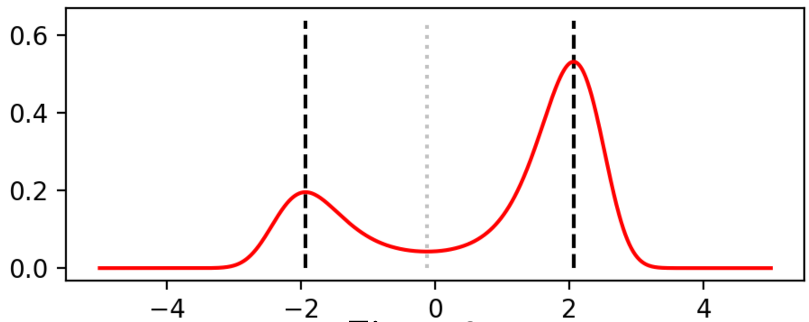

Figure 8

Solution to the stationary Fokker-Planck equation for $\alpha = 0.5$ $\beta = 4$ $\sigma = 2$ for the potential function of the cusp. Black dashed lines indicate stable equilibria, silver dotted line indicates unstable equilibrium.

In ordinary regression the predicted value is equal to the expected value of the dependent variable given values of independent variables. Consider a linear regression; the model predicts a single value along the fitted line, with some normally distributed error around such a value. The cusp is not a regular regression model as for certain combinations of α and β (inside the bifurcation region) the system has multiple stable states. This means that the model prediction is not a single value, but a range of values with a bimodal probability density as shown in Figure 8. Therefore, if it is necessary to derive a single value, a convention to select such expected values needs to be adopted. There are two conventions at philosophical extremes:

- **Delay convention:** Choose the mode of the probability density closest to the current state as the predicted value. Cobb (1981) recommends this convention and (Grasman et al., 2009) implement this as the default in their software package.

- **Maxwell convention:** Choose the mode at which the probability density is highest.

Both conventions are valid, and can be chosen by the researcher depending on theoretical reasons. The delay convention predicts the mode of the density closest to the current state, e.g. the value that the current state is most likely to converge to in the near future. The Maxwell convention predicts the mode of the density with the highest probability, e.g. where the long-run system state spends the most time in.

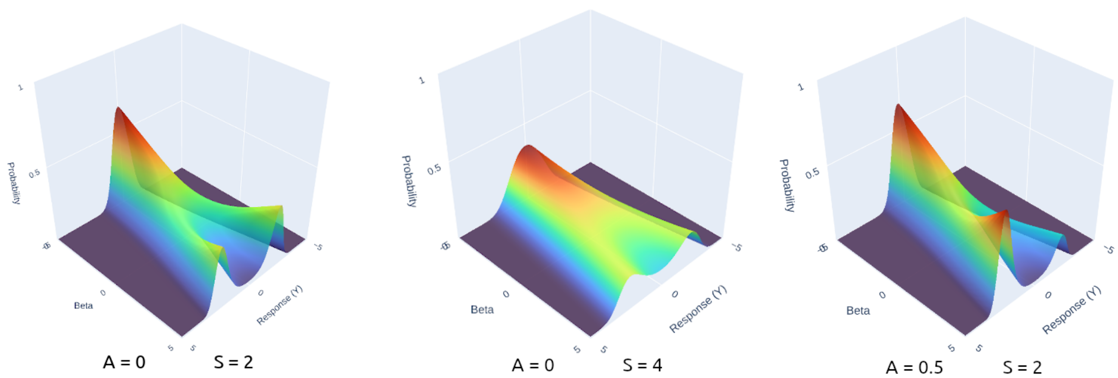

Figure 9

Probability densities for the cusp (1) $\alpha = 0$, $\sigma = 2$; (2) $\alpha = 0$, $\sigma = 4$; (3) $\alpha = 0.5$, $\sigma = 2$. Increasing σ distributes the density across a wider range of values. Varying β can cause a bifurcation. Varying α determines the relative density of the two modes in a bifurcation.

Grasman et al. (2009) published a package in the statistical software R that implements Cobb's method, augmented with the multivariate technique of Oliva et al. (1987) which allows for a set of dependent variables to be embedded into a single control variable as a linear combination. In addition the package implements the suggestions and improvements of A. Hartelman (1998) to make the algorithm more robust. However, it is only suitable for cross-sectional data.

Grasman's package makes the assumption that the control and state variables observed in the data are unknown smooth transformations of the canonical control and state variables. In particular, the package assumes the simplest and most straightforward case - that this transformation between observed and canonical units is a linear one. To understand this, one can imagine recording data from a cusp that is stretched and warped -

the upper and lower folds could slope much more drastically, but the underlying bifurcation structure and qualitative system behavior remains the same. The package therefore attempts to find the linear coefficients that would transform the observed variables (stretched under some unknown transformation) into canonical ones.

Assuming the data is appropriate for a cusp model and the fitting procedure retrieves the correct linear coefficients, we can use these coefficients to transform the data into canonical units and apply the Maxwell or Delay conventions to make long-run predictions. We can also use these coefficients, by inverting the line equation to express, for instance, Cardan's discriminant in the original units. This would give us the boundary of the fitted bifurcation area in the original units. This information could, assuming a valid model, be used to predict and make statements about critical transitions and hysteresis in the system.

Example Stochastic Cusp Model Fit

This section introduces a sample fitting procedure for the cusp using simulated data. The data have intentionally been made highly noisy to demonstrate what results can be expected with real-world data.

Generating Sample Cusp Data

In order to generate 100 cross-sectional observations, 100 points are uniformly sampled at random from the cusp control plane in the range $-2 < \alpha < 2$ and $-2 < \beta < 2$. Each sampled point is given an initial condition x_0. Each α, β, x_0 triplet serves as an initial condition for stochastic differential equation 12 and is allowed to evolve over time for 500 time-steps with $\sigma = 1$ and $dt = 0.01$ creating 100 trajectories in time-series. For each trajectory the initial condition x_0 is set equal to the stable equilibrium point for current α, β perturbed by a small random amount when there is only one equilibrium (i.e. the point begins outside of the bifurcation area), and the unstable equilibrium perturbed by a small random amount when there are three equilibria. At each time step the control parameters α and β for each trajectory are randomly perturbed by a small amount (i.e. the points are walking randomly along the cusp surface). An example trajectory for a single initial condition is illustrated in Figure 10.

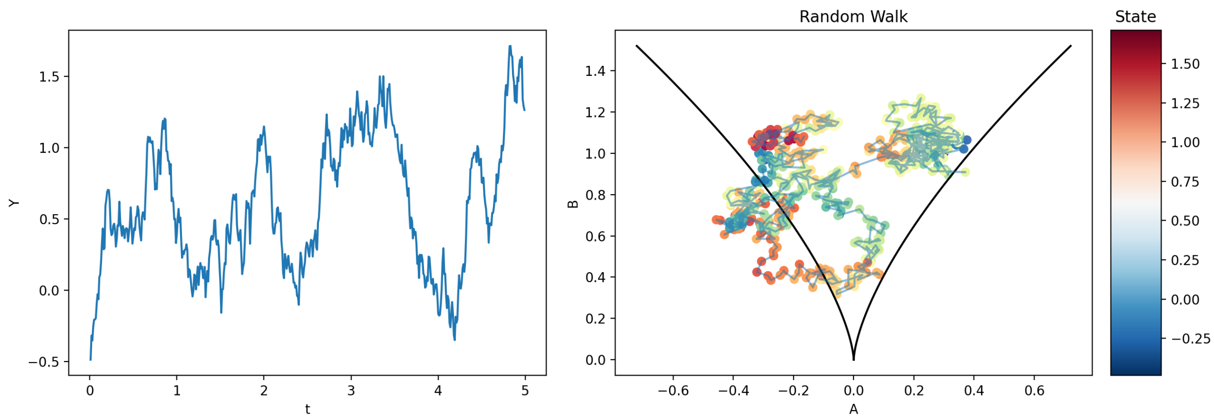

Figure 10

Left: Sample trajectory with initial condition $\alpha = 0.3$, $\beta = 1$ and x_0 equal to the corresponding unstable equilibrium perturbed slightly. Right: random walk path taken on the control surface. The α, β, and y are recorded at the final time-step yielding a single cross-sectional observation.

Finally, we sample the α, β and final state x at the end of each trajectory to yield 100 cross-sectional observations. The final data-set can be seen in Figure 11 and found in Appendix A.

Fitting Sample Data Using Cobb's Method

As seen in Figure 10 the time-series of each trajectory can show sudden transitions due to stability loss or noise perturbing the system state sufficiently close enough to another equilibrium. However, as mentioned above, Cobb's method does not allow us to say anything about these intra-individual differences or individual time-dependent evolution. Instead, we capture a cross-section of each trajectory and make the assumption that the underlying system dynamics are the same for each sampled point.

We now use the R package 'cusp' (Grasman et al., 2009) to fit the sample dataset to the cusp catastrophe, yielding the coefficients in table 2, and model comparison statistics in

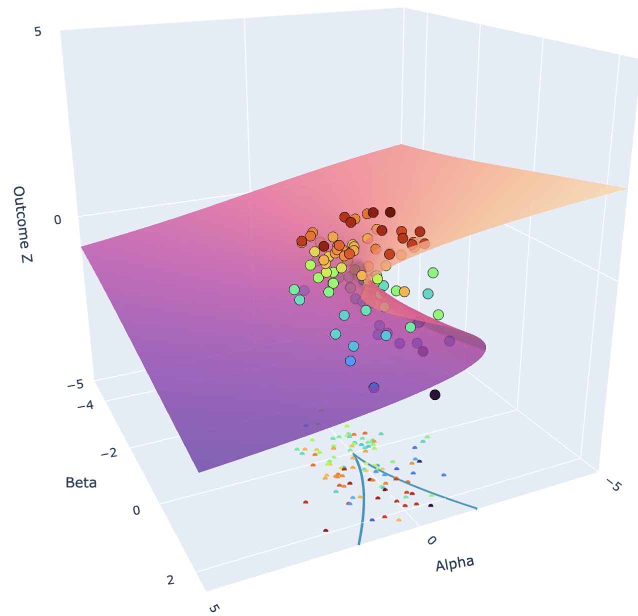

Figure 11

Sample noisy cross-sectional dataset (Appendix A) generated from the cusp, with an overlay of the cusp surface.

table 3.

Table 2 provides linear coefficients to transform observed control variables α and β into canonical coordinates, and behavioral variable y into the the corresponding location and scale. Notice that some coefficients are not significant - this is expected as we did not add units (i.e. we didn't transform the location and scale of the observations). Figure 12 shows the original data in figure 11 transformed into canonical variables using the coefficients found by the fitting procedure in table 2.

Table 3 provides model comparison statistics. The R^2 value is in fact a $pseudo - R^2$ statistic that can become negative (due to the relative skew of the density) and should not be compared to the R^2 statistics in regular regression models. The $pseudo - R^2$ is difficult to interpret, therefore it is preferable to select models using log-likelihood, AIC, and BIC (Grasman et al., 2009). The log-likelihood, AIC, and BIC all indicate the cusp as the best model to explain the given data.

Table 2

Fitted coefficients

	Estimate	**Std. Error**	**z-score**	**p-value**
a[(Int)]	0.13986	0.12574	1.112	0.26601
a[a]	0.46657	0.15085	3.093	0.00198 **
b[(Int)]	0.20614	0.47138	0.437	0.66189
b[b]	1.61273	0.27619	5.839	5.24e-09 ***
w[(Int)]	0.10133	0.08026	1.262	0.20678
w[y]	1.17482	0.07992	14.700	$<$ 2e-16 ***

Table 3

Model comparison

	R.Squared	**logLik**	**npar**	**AIC**	**AICc**	**BIC**
Linear model	0.1907656	-127.0183	4	262.0365	262.4576	272.4572
Logist model	0.1856229	-127.3350	5	264.6700	265.3083	277.6959
Cusp model	0.6523225	-109.5451	6	231.0902	231.9934	246.7212

Inference

The fitting procedure has attempted to find the linear transformation that would transform the observed control coordinates α, β and behavioral variable x into canonical

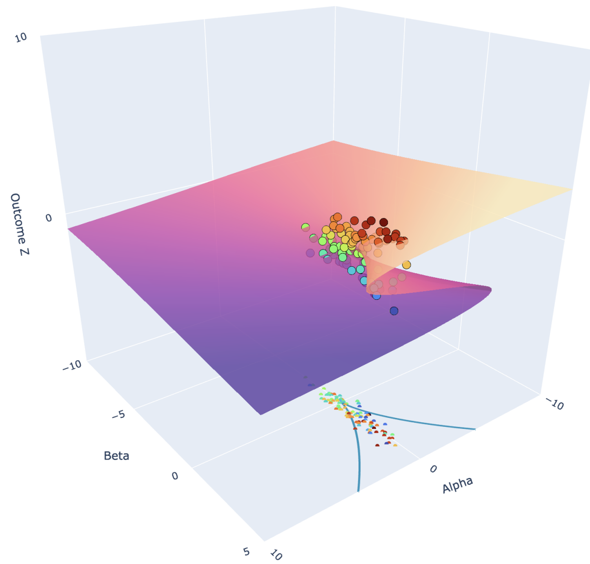

Figure 12

Original sample dataset transformed into estimated canonical units using linear coefficients retrieved via Cobb's method. The location and scale of the data are a slightly off from the original data (see Figure 11). This is not surprising as the noise level during sampling was very high given the scale of α and β.

units. Finally, with the data transformed into canonical units we use the Delay convention to predict the long-run most-likely state for each of the cross-sectional points. As discussed above, the delay convention selects the mode of the probability density closest to the current state. Figure 13 shows the cusp regression predictions for the sample dataset. The prediction outputs the converged long-run most likely value for the α, β and y the cross-sectional point was observed at, which will coincide with an equilibrium on the cusp surface.

Since we sampled the data from the cusp, we can compare the ground truth values to model estimates. Figure 14 shows how well the fitted model classifies the observations as inside or outside of the bifurcation area.

Working strictly with cross-sectional data creates strong assumptions and limitations. Firstly, we must assume that each observation is governed by the same cusp

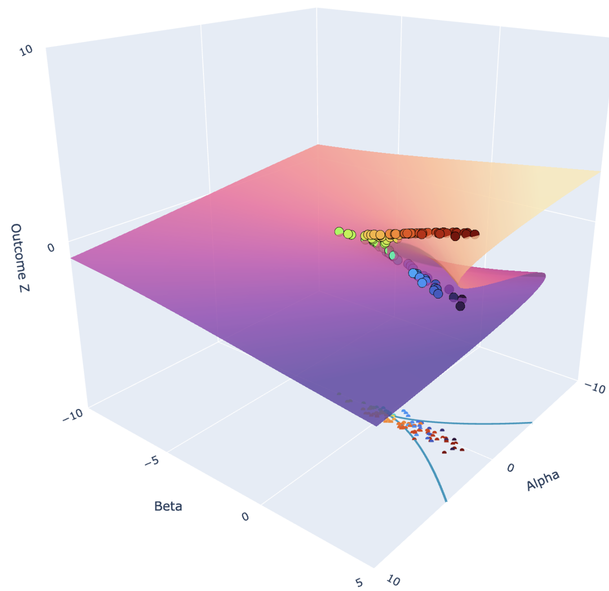

Figure 13

Predicted values of cusp regression using the delay convention. Each data point is assumed to have converged to the closest mode of the probability density.

dynamics (i.e. the potential function or cusp surface does not substantially differ for individual observations). Therefore the model always reflects between-individual behavior; the individuals are assumed to behave homogeneously. Secondly, because this cusp regression is time independent and cannot capture individual dynamics, inference and interpretation is limited. The regression selects the most likely long-run converged value (depending on chosen convention) *only* for the current provided α, β and y values. Therefore, we cannot make any statements or predictions that involve time. It is difficult to think of situations outside of the physical sciences and particularly in the social sciences where such predictions would be meaningful or useful.

Applications of the Stochastic Cusp

Stochastic CT has seen various application attempts: in modeling the trade-off between speed an accuracy in reaction time experiments (Dutilh et al., 2011), crash rates in urban arterial roads (Park & Abdel-Aty, 2011), withdrawal in construction project

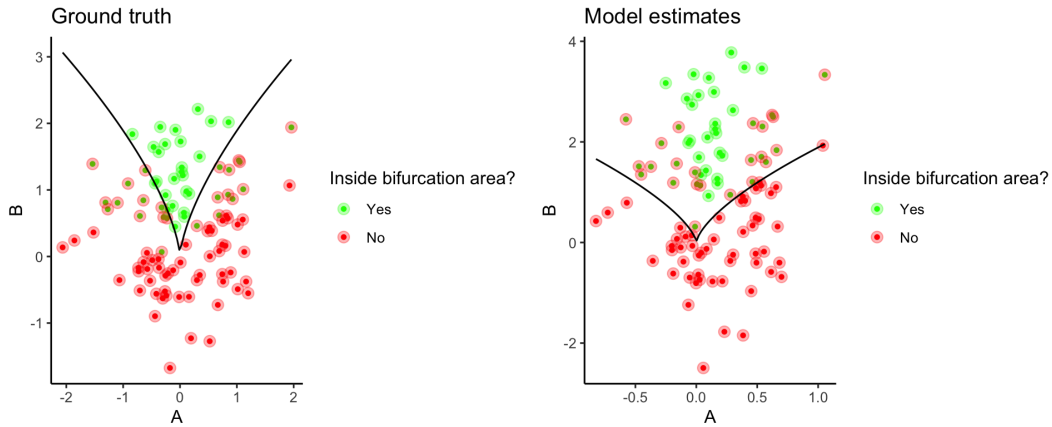

Figure 14

Left: the control surface with the ground truth bifurcation boundary. Right: the bifurcation boundary from the perspective of the fitted model. Large points are the ground truth, small points are model predictions. Green indicates the datapoint being inside the bifurcation area, red indicates the point being outside of it. Therefore, small and large points that have the same color indicate correct classification by the model, and vice versa.

negotiations (P. T. Chow et al., 2012), cheating behavior in students with learning disabilities (Sideridis & Stamovlasis, 2014), sudden transition of the housing market in Poland (Bełej & Kulesza, 2013), driver aggression in freeway traffic (Papacharalampous & Vlahogianni, 2014), grip strength (D. G. D. Chen et al., 2014), and absenteeism in firefighter teams (Marques-Quinteiro et al., 2020).

Current social science applications tend to have one or more of the following limitations. First, low sample sizes. Hartelman's (P. A. I. Hartelman et al., 1998) simulations show it takes hundreds of observations to reliably recover the parameters using Cobb's method. Even with sufficient sample sizes, Hartelman demonstrates that estimators are biased. Unfortunately, publications often have inexcusably low sample sizes. Second, indirectly observed variables with convoluted ad-hoc operationalization, arbitrary scalings, magic numbers and arbitrary exclusion criteria. Many variables of interest in the social sciences are not directly observable and / or difficult to quantify; however, variable construction should be grounded in theory and carefully explained. Third, using time-series data to fit a cusp model using Cobb's method. As discussed above, Cobb's method is not suitable for time-series data and in the overwhelming majority of cases the model will fail to converge properly with longitudinal or time-series data. This is due to the difficulty of obtaining a time-dependent solution for stochastic differential equation 12. Therefore, intra-individual differences and transitions can not be modeled using Cobb's method in it's current form. Fourth, only comparing cusp model fit to linear model fit, when comparison to a logistic model is available. Comparing a fitted cusp model to a logistic model is important, as the logistic model can approximate arbitrarily steep transitions, but can not explain hysteresis. If the data contains steep or discontinuous transitions, the cusp will almost always be selected over the linear model, but that does not guarantee that a cusp model is in fact the best explanation of the data. The logistic model is an important intermediary step. Fifth, disappointingly limited conclusions, e.g. 'phenomena x is explained by the cusp'. The overwhelming majority of papers outside of engineering and physical applications seem to stop at model fit comparisons, just short of making any quantitative predictions or guidelines for real-world utility. Finally, sixth - concluding that the data are better explained by the cusp model (given all the above points) when the AIC and BIC beat linear and logistic models by just a few points.

Discussion

Despite being the standard approach, Cobb's method comes with obstacles and limitations that are important to recognize before attempting to fit or interpret a cusp model in a social science context.

Current easily accessible implementations of Cobb's method are only suitable for cross-sectional data and cannot be applied to time-series or longitudinal data, leaving out the heart of the dynamical systems approach. Unfortunately, this fact hasn't stopped some researchers from attempting to fit time-series data to the cusp using this approach anyway.

Cobb's method makes the assumption that the control and state variables are continuous. In the social sciences variables are often measured in Likert-scales, and similar ordinal measures; Grasman et al. (2009) appropriately handle and provide an example where they fit the model on ordinal data, but the potential consequences of ignoring this assumption have not yet been discussed critically. Of course, where true continuous data is unavailable, ordinal data are better than nothing, but caution should be exercised.

Using simulated data (sampled directly from the cusp) we found that the fitting procedure recovers the parameters reasonably well when the cross-sectional data are rather uniformly distributed around the origin of the control surface (or on one side of $a = 0$ on the control plane) and the noise is not too high. However, the fit can sometimes inexplicably converge to very different and incorrect solutions, particularly (but not necessarily) when the data show a less uniform pattern or are patchy. Before the fit, the researcher must select initial parameter values, which can yield very different converged solutions depending on this choice. In some cases, even providing the correct proximal starting values in a high sample size generated dataset will not help the fitting procedure converge to a correct or even sensible solution. Given that the control variables are often in practice chosen ad-hoc and the model is itself not very transparent, this places a burden on the researcher to carefully evaluate whether the fitted model has converged somewhat correctly.

Because time does not play any role in the fitted model, the predicted value(s), in most cases, have very limited utility and interpretation. The predicted values make the assumption that the observation has converged to the most likely state (the most likely equilibrium decided by applying the Maxwell, or Delay conventions). As time is not involved, it's difficult to interpret what such a prediction means or how and if it can be effectively used in, for instance, clinical applications.

Catastrophe models were popularized using intuitive, conceptual dynamical systems examples demonstrating sudden behavioral transitions (such as fear and rage in dogs) along with the promise of anticipating these transitions. However, currently available methodology cannot deal with (noisy and difficult to measure) time-series or longitudinal data. Therefore, the intuitive conceptual examples on which CT became popular cannot, in most cases, be implemented in practice given current limitations. Moreover, the necessary time investment for a social scientist to sufficiently understand stochastic catastrophe theory and it's current methodological nuances is high, therefore the chance of misunderstandings is also high. A combination of 'researcher degrees of freedom' and current methodological limitations make the rigorous application of stochastic catastrophe theory a difficult and a niche case in the social sciences.

This does not imply that catastrophe theory or the discussed methodology do not have a place in social science applications. On the contrary, CT is a powerful, very promising tool that with careful application can explain otherwise impossible to model phenomena. In large part, the issues of successful application lie in data-model fit.

The usage of catastrophe models in the social sciences would benefit from a better understanding of current methodological limitations and a systematic way of handling data, fitting the model, checking model validity, comparing competing models, and predicting outcomes. There is an explanatory gap between the observed data and fitted (time-independent) model predictions that needs to be filled. In addition, methodological developments that could accommodate time-series and longitudinal data in a way that is approachable to most researchers would target far more social science applications and offer more utility. This is a difficult problem, but promising work is being done.
S. M. Chow et al. (2015) have attempted to accommodate longitudinal data using mixed structural equation modeling with regime switching (MSEM-RS) in a way that mimics the rudimentary features of the cusp catastrophe. However, the relationship of this approach to catastrophe theory is only inspirational, and comes with it's own list of assumptions and limitations. D. G. Chen et al. (2021) propose a novel way of using Bayesian inference to estimate parameters in the cusp.

In current form, many attempts to apply catastrophe theory to the social sciences are of limited scope, difficult to interpret, don't explicitly make predictions, and are unlikely to replicate.

This thesis comes with a set of limitations. Firstly, the lack of author expertise on the subject. Secondly, the pool of catastrophe theory applications in the social sciences is relatively small and is overshadowed by applications in the physical and financial sciences. In addition, these few application attempts usually involve very specific and niche behavioral phenomena. It is therefore difficult to judge the ability of current CT methodology to generate replicable predictions on large and rich datasets in a clinical context. Finally, this thesis highlights common issues, misconceptions, and limitations, but does not propose a solution to overcome them.

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Appendix A: Sample Generated Cross-sectional Cusp Data

	alpha	beta	y
0	0.743680453	-0.26557761	-0.228586503
1	0.925098921	0.864681717	1.114504518
2	0.863389794	1.302246092	-1.061078475
3	0.297221947	0.460233727	0.70660262
4	-0.288797784	0.591443555	-1.191545742
5	0.690171896	0.081943274	-0.425020859
6	0.891217086	-0.239812509	0.931980734
7	0.347715786	-0.278547959	0.721643075
8	-0.08545891	0.447445998	-0.080316885
9	-0.103262127	1.168512129	0.901632645
10	-0.240001212	-0.587280559	-0.120793827
11	-0.321769694	0.067438715	0.725434368
12	-0.574187634	-0.188028111	-0.526230606
13	-1.270300022	0.710399074	-1.218059711
14	0.48785604	0.38093078	-0.893831117
15	-0.610154449	1.295293043	1.006899578
16	0.663705725	0.889074643	1.609691984
17	-0.130448567	0.760067088	-1.298423837
18	0.687085126	0.619153977	0.863111082
19	0.854071996	2.017448108	-1.623526637
20	0.048897397	1.220324963	-0.318258029
21	0.559103949	0.386385281	0.298283914
22	1.960181129	1.94020396	1.739759952
23	0.158172816	0.942577231	1.781236588
24	-0.253631459	0.922115742	-1.188658204
25	0.075202755	0.59932144	-0.731899155
26	0.007861049	-0.090356996	-0.538429251
27	0.749386939	0.538942854	0.937453259
28	1.016343286	-0.490250883	0.596220037
29	-0.488854341	-0.055086542	-1.006923887
30	-1.098476435	0.806895456	0.823107093
31	-0.304233505	-0.628883401	-0.572315383
32	1.050949844	1.421706032	1.192081885
33	1.162743854	-0.376477761	-0.772543075
34	-0.837537604	1.837904311	-0.35160397
35	-0.731403576	-0.170299578	0.789888773
36	0.297960696	-0.354898178	-0.210551834
37	-1.537978149	1.390726207	-1.389312333
38	1.013472663	0.479738136	0.527822507
39	0.845873212	0.928497496	0.88476264
40	1.034473312	1.443515171	-1.296251016
41	-0.369195921	-0.169100996	-0.991357758
42	-0.727274876	-0.221787101	-0.587471848
43	-0.706116804	-0.511052459	1.041191492
44	0.030989332	1.268742775	-0.996092397
45	-1.0654763	-0.354220559	-0.802997765

46	-0.120352867	-0.206472456	0.483286619
47	1.104461512	0.554986647	0.659984073
48	0.518033708	0.439026778	1.525857156
49	-0.704872337	0.608211572	-0.386132167
50	-0.264983244	-0.523749822	-0.78580599
51	-0.015129788	-0.607898811	-0.530247664
52	0.032356676	1.337589197	1.92666942
53	-0.347627763	1.946191801	1.424525292
54	-0.214971635	-0.255326489	-0.188322203
55	-0.465093267	1.645456365	1.542704126
56	0.103119012	0.177282906	0.641664956
57	-0.17649139	-1.674367035	-0.104352856
58	1.130148524	0.069203374	1.017127875
59	0.816090159	0.612415389	0.881004513
60	0.524751192	0.003747898	-0.274186533
61	-0.26231568	1.688191093	-0.616097598
62	-0.255095447	-0.28714583	-0.967221498
63	-0.437972886	1.100616982	0.76960351
64	-1.858061508	0.241033276	-0.864336873
65	-0.217326342	0.703716008	-0.333082784
66	-0.914108014	1.096274836	-1.482123556
67	0.840130645	0.574911357	0.789384002
68	-2.064906427	0.136660349	-0.403235083
69	0.069288384	0.65714621	0.380386418
70	0.758325371	-0.377206953	0.379237003
71	0.315060522	2.214374162	0.599670721
72	0.157734519	-0.605167813	-0.647606378
73	1.926751905	1.067522934	1.462059069
74	-0.374286654	1.571005657	1.287845731
75	0.79809118	0.161520862	0.123993651
76	1.198988904	-0.551727493	-0.590889517
77	1.111572272	1.010735124	-0.88205547
78	-0.439479288	-0.896537482	-0.117084071
79	-1.310349957	0.8109599	-1.950087213
80	-0.644842972	0.844278604	-1.580881994
81	-1.522422881	0.362140147	-1.49940408
82	-0.323432245	0.73682651	0.36544752
83	0.193868818	-1.22813445	-0.349925892
84	-0.416367416	-0.560268889	0.936655132
85	0.010551326	1.727894294	1.507437579
86	-0.079354863	1.902485084	1.427416325
87	-0.412948268	1.127981506	1.338459368
88	-0.640502737	-0.083421339	-0.335441256
89	0.744596824	0.178127394	0.506602698
90	0.112515767	0.978012625	0.945318185
91	-0.229079864	0.575206468	-0.019738865
92	-0.526549881	-0.364030922	-0.569051776
93	0.546130341	2.031748681	1.374843669
94	0.698817101	1.341670343	0.602336027
95	0.523002259	-1.272968543	-1.104635392
96	-0.378868949	-0.03962228	-0.743599222
97	0.344601798	1.502829874	1.635724517
98	0.665913807	-0.727264191	-0.139660154
99	-0.581607413	0.054602164	-0.254823906